III

APPLICATION TO THE PENDULUM WITH AN OSCILLATING SUPPORT

30. The first problem which we shall consider is the motion of an ordinary pendulum which is subjected to a type of disturbance liable to produce resonance effects.

For simplicity, we shall suppose that the pendulum is the ideal simple one of length b. The disturbance is communicated through the point of support S which is supposed to be movable in a horizontal direction only.

Let y be the horizontal distance of S from a fixed point O and x the angle which the pendulum makes with the

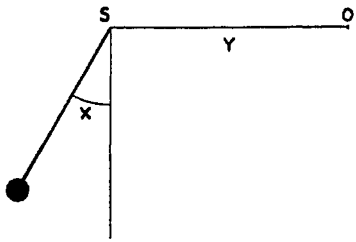

FIG. I

vertical at time t. The equation of motion of the pendulum is then

$$\frac{d^2y}{dt^2}\cos x + b\,\frac{d^2x}{dt^2} = -g\sin x. \tag{30.1}$$

Suppose that S is forced to oscillate with a motion defined by

$$y = -b'\sin(n't+\epsilon'), \tag{30.2}$$

where b', n', ϵ' are given. It is thus supposed to be unaffected by the oscillation of the pendulum.

With this value for y, equation (30.1) becomes

$$
\begin{aligned}
b\,\frac{d^2x}{dt^2}+g\sin x &= -n'^2b'\sin(n't+\epsilon')\cos x \\
&= -\,n'^2b'\,\frac{\partial}{\partial x}\left\{\sin(n't+\epsilon')\sin x\right\}. \qquad (30.3)
\end{aligned}
$$

If we put

(30.4) $g/b=\kappa^2$, $n'^2b'=\kappa^2mb$, $\phi=-\kappa^2\sin(n't+\epsilon')\sin x$,

equation (30.3) may be written

$$\frac{d^2x}{dt^2}+\kappa^2\sin x=m\,\frac{\partial\phi}{\partial x}\,. \qquad (30.5)$$

31. In order to solve (30.5) according to the method developed above, we first solve the equation when $m=0$. The solution has already been obtained. As we shall suppose that the pendulum is performing oscillations with not very large amplitude, the solution is that given in section 16, namely,

$$
\left\{
\begin{aligned}
x &= c\sin l+\frac{c^3}{192}\sin 3l+\cdots \\
n &= \kappa\left(1-\frac{c^2}{16}+\cdots\right),\quad \frac{dx}{dt}=n\,\frac{\partial x}{\partial l}\,,
\end{aligned}
\right. \qquad (31.1)
$$

these values satisfying

$$n^2\frac{\partial^2x}{\partial l^2}+\kappa^2\sin x=0 \qquad (31.2)$$

for all values of c, l.

When m is not 0, we choose c, l as the new variables and, according to (25.1), they satisfy the equations

$$\frac{dc}{dt}=\frac{m\partial\phi}{K\partial l}\,,\quad \frac{dl}{dt}=n-\frac{m\partial\phi}{K\partial c} \qquad (31.3)$$

where K is given by (24.7).

The calculation of K is quite easy if we make use of the fact that it is independent of l. Let us put $l=90°$ after forming the derivatives from (31.1). The first term of (24.7) vanishes and for the second term we have, when $l=90°$,

$$\frac{\partial x}{\partial c}=1-\frac{c^2}{64}+\cdots, \quad \frac{\partial^2 x}{\partial l^2}=-c+\frac{3c^3}{64}\cdots,$$

so that

$$K=\kappa c\left(1-\frac{c^2}{8}+\cdots\right). \tag{31.4}$$

Next, ϕ has to be expressed in terms of c, l. The simplest way to do this is to calculate $\sin x$ from (31.2) with the help of (31.1). We find

$$\sin x=-\frac{n^2\partial^2 x}{\kappa^2\partial l^2}=\frac{n^2}{\kappa^2}\left(c\sin l+\frac{3c^3}{64}\sin 3l+\cdots\right).$$

Hence, from (30.4),

$$\begin{cases}\phi=-\frac{1}{2}n^2c\cos(l-n't-\epsilon')+\frac{1}{2}n^2c\cos(l+n't+\epsilon')\\ \qquad -\frac{3}{128}n^2c^3\cos(3l-n't-\epsilon')\\ \qquad +\frac{3}{128}n^2c^3\cos(3l+n't+\epsilon')+\cdots.\end{cases} \tag{31.5}$$

The substitution of this value of ϕ in (31.3) gives the required equations for finding c, l.

32. Since m is supposed to be small and since we shall neglect m^2, the various terms in ϕ may be regarded as separate disturbances, each producing its own effect: the total effect being the sum of the separate portions in this approximation.[1]

The chief interest centers on the first term because it gives resonance phenomena when $n=n'$. For this term the solu-

[1] The method of Delaunay, as used in celestial mechanics, avoids this assumption.

tion in sections 26–29 may be directly used. We have, in fact,

$$l_i = l - n't - \epsilon', \; i = 1, \; a_i = \tfrac{1}{2}n^2 c.$$

Also, from (31.1),

$$\frac{dn}{dc} = -\tfrac{1}{8}\kappa c + \cdots.$$

Following the notation of these sections, we have $n_0 = n'$, so that by (31.1)

$$c_0 = 4\left(\frac{\kappa - n'}{\kappa}\right)^{\frac{1}{2}}, \tag{32.1}$$

approximately.

Hence a first approximation shows that an oscillation of the pendulum with frequency n'—the same as that of the disturbing force—is possible provided the arc through which it oscillates is given by (32.1). Evidently, it is necessary that $\kappa > n'$, since this arc must be greater than zero in order that motion shall occur.

This last condition exhibits the necessity for dealing with the non-linear equation for the motion of the pendulum. If we had used the linear form, the frequency n_0 would have been put equal to κ, and there would have been no clue to the value of the amplitude under resonance conditions.

The frequency p of a small oscillation about this resonance configuration is given by (27.4). In the present case it is approximately given by

$$p^2 = \left| \frac{m}{\kappa c} \quad \frac{\kappa c}{8} \cdot \tfrac{1}{2}\kappa^2 c \right|_0 = \frac{m\kappa^2 c_0}{16}.$$

Hence we have, from (32.1),

$$p = \frac{m^{\frac{1}{2}}\kappa}{2}\left(\frac{\kappa - n'}{\kappa}\right)^{\frac{1}{4}}.$$

The small oscillation of l is given by

$$l = n't + \epsilon' + \lambda \sin(pt + \lambda_0),$$

and that of c by (section 28)

$$c = c_0 - \frac{ma_1}{Kp} \lambda \cos(pt + \lambda_0),$$

where c_0 has the value (32.1). This last coefficient is

$$\frac{m\frac{1}{2}\kappa^2 c_0}{\kappa c_0} \cdot \frac{2}{m^{\frac{1}{2}}\kappa} \cdot \left(\frac{\kappa}{\kappa - n'}\right)^{\frac{1}{2}} \lambda = m^{\frac{1}{2}} \left(\frac{\kappa}{\kappa - n'}\right)^{\frac{1}{2}} \lambda.$$

The maximum amplitude of oscillation of c is given by (29.1). This maximum is found to be

$$2m^{\frac{1}{2}} \frac{n'}{\kappa} \left(\frac{\kappa}{\kappa - n'}\right)^{\frac{1}{2}}.$$

In order that the approximations may be valid, it is evidently necessary that $\kappa - n'$ shall not be too small.

33. When $\kappa = n'$, the method adopted above fails, and in this case it appears to be easier to return to the original equation (30.3). This, with the notation (30.4), can be written

$$\frac{d^2x}{dt^2} + \kappa^2 \sin x + \kappa^2 m \cos x \sin(\kappa t + \epsilon') = 0. \tag{33.1}$$

When x is small it is possible to expand $\sin x$, $\cos x$ in powers of x. Let us see whether a solution with period $2\pi/\kappa$ can be found; if so, only odd multiples of $\kappa t + \epsilon'$ will be present. Suppose

$$x = A_1 \sin(\kappa t + \epsilon') + A_3 \sin 3(\kappa t + \epsilon') + \cdots,$$

where A_1, A_3, are small. We have

$$\cos x = 1 - \tfrac{1}{2}x^2 + \cdots = 1 - \tfrac{1}{4}A_1^2 + \tfrac{1}{4}A_1^2 \cos 2(\kappa t + \epsilon') + \cdots$$

$$\sin x = x - \tfrac{1}{6}x^3 + \cdots = A_1 \sin(\kappa t + \epsilon') + A_3 \sin 3(\kappa t + \epsilon') + \cdots$$
$$- \tfrac{1}{8}A_1^3 \sin(\kappa t + \epsilon') + \tfrac{1}{24} A_1^3 \sin 3(\kappa t + \epsilon') + \cdots.$$

Inserting these in (33.1) and equating to zero the coefficients of $\sin(\kappa t + \epsilon')$, $\sin 3(\kappa t + \epsilon')$, we obtain

$$-A_1\kappa^2 + \kappa^2(A_1 - \tfrac{1}{8}A_1^3) + m\kappa^2(1 - \tfrac{3}{8}A_1^2) + \cdots = 0,$$

$$-9A_3\kappa^2 + \kappa^2(A_3 + \tfrac{1}{24}A_1^3) + \tfrac{1}{8}m\kappa^2 A_1^2 + \cdots = 0.$$

The first approximation to the solution of these equations gives

$$A_1 = 2m^{1/3}, \qquad A_3 = \tfrac{1}{192} A_1{}^3 = \frac{m}{24}\,.$$

It is evident that the coefficient of A_i will have $m^{i/3}$ as a factor, that the approximations proceed along powers of $m^{2/3}$, and that the series converge.

Thus a solution which depends on $m^{1/3}$ exists. This, however, is a *particular* solution since it contains no arbitrary constants. Let us call it $x = x_0$, and see whether a solution $x = x_0 + \delta x$ can be found. On inserting this value of x in (33.1) and neglecting powers of δx beyond the first, we obtain

$$\frac{d^2}{dt^2}\,\delta x + \{\kappa^2 \cos x_0 - \kappa^2 m \sin x_0 \sin(\kappa t + \epsilon')\}\,\delta x = 0.$$

The principal terms in the coefficient of δx are those deduced by putting

$$\cos x_0 = 1 - \tfrac{1}{2}x_0{}^2 = 1 - \tfrac{1}{4}A_1{}^2 + \tfrac{1}{4}A_1{}^2 \cos 2(\kappa t + \epsilon'),$$

where $A_1 = 2m^{1/3}$. In general, the coefficient of δx will be a series of the form,

$$1 + \sum_{0}^{\infty} b_{2i} \cos 2i(\kappa t + c'),$$

where b_{2i} vanishes with m and b_0 is not zero.

Equations of this type are well known. They give oscillatory motion with an arbitrary amplitude and phase. The principal frequency present depends on b_0. Thus small variations from the particular solution appear to be possible.

34. Hence when $\kappa - n'$ is zero, the resonance case gives expansions in powers of $m^{1/3}$, while we saw in the first part, when $\kappa - n'$ was not too small, that the expansions proceeded along powers of $m^{1/2}$. The theory of analytic forms suggests that a singularity may separate these two sets of solutions.

The method of investigation followed in section 33 might also have been used in the earlier case where $\kappa - n'$ was not zero. It is difficult, however, to get a clue with this method to the transition from non-resonance to resonance and, in any case, it is much less adaptable to the more complicated cases presented in other problems.

Another exercise which may be left to the student is the investigation of the resonances arising when one of the relations $3n - n' = 0$, $5n - n' = 0, \cdots$, is approximately satisfied. These resonances require the consideration of the terms with arguments $3l - n't - \epsilon', \cdots$, in (31.5). It can be shown that as long as we retain only the lowest power of m present, each can be treated as though the remaining terms did not exist.